On Linear Differential Equations whose Fundamental Integrals are the Successive Derivatives of the same Function.

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I.

It is known that having given a linear differential equation of the nth order, one of whose fundamental integrals is a function of another, the equation can be reduced to one of the order $n-1$. For example, having. given the linear differential equation

1. $$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_n y = 0,$$

of which the fundamental (*i. e.* linearly independent) integrals are

$$y_1, y_2, y_3 \ldots y_n;$$

then if $y_i = \phi(y_j)$, equation 1 can be reduced to the form

$$\frac{d^{n-1} y}{dx^{n-1}} + q_1 \frac{d^{n-2} y}{dx^{n-2}} + q_2 \frac{d^{n-3} y}{dx^{n-3}} + \ldots + q_{n-1} y = 0,$$

where $q_1, q_2, q_3 \ldots q_{n-1}$ are known functions of $p_1, p_2, p_3 \ldots p_n$.

Assuming now the equation

2. $$\frac{d^n y}{dx^n} + p_{11} \frac{d^{n-1} y}{dx^{n-1}} + p_{12} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_{1n} y = 0,$$

having for integrals $\quad \phi, \dfrac{d\varphi}{dx}, \dfrac{d^2\varphi}{dx^2} \ldots \dfrac{d^{n-1}\varphi}{dx^{n-1}},$

it is required to find the form of the coefficients $p_{11}, p_{12} \ldots p_{1n}$. We have to start with the following system of equations

3.

$1_0 \qquad \dfrac{d^n \varphi}{dx^n} \quad + p_{11} \dfrac{d^{n-1}\varphi}{dx^{n-1}} + \ldots + p_{1n} \phi \quad = 0$

$1_1 \qquad \dfrac{d^{n+1}\varphi}{dx^{n+1}} + p_{11} \dfrac{d^n \varphi}{dx^n} \quad + \ldots + p_{1n} \dfrac{d\varphi}{dx} \quad = 0$

$1_2 \qquad \dfrac{d^{n+2}\varphi}{dx^{n+2}} + p_{11} \dfrac{d^{n+1}\varphi}{dx^{n+1}} + \ldots + p_{1n} \dfrac{d^2\varphi}{dx^2} \quad = 0$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$1_{n-1} \quad \dfrac{d^{2n-1}\varphi}{dx^{2n-1}} + p_{11} \dfrac{d^{2n-2}\varphi}{dx^{2n-2}} + \ldots + p_{1n} \dfrac{d^{n-1}\varphi}{dx^{n-1}} = 0.$

Differentiating the first of these and subtracting the result from the second we have, using accents to denote differentiation,

4. $$p'_{11}\phi^{(n-1)} + p'_{12}\phi^{(n-2)} + \ldots + p'_{1n}\phi = 0,$$

one integral of which is of course the function ϕ. Differentiating now the second of equations 3 and subtracting the result from the third, then differentiating the third and subtracting from the fourth, and so on, we have the equations

$$p'_{11}\phi^{(n)} \quad + p'_{12}\phi^{(n-1)} + \ldots + p'_{1n}\phi' \quad = 0$$
$$p'_{11}\phi^{(n+1)} + p'_{12}\phi^{(n)} \quad + \ldots + p'_{1n}\phi'' \quad = 0$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$p'_{11}\phi^{(2n-3)} + p'_{12}\phi^{(2n-4)} + \ldots + p'_{1n}\phi^{(n-2)} = 0,$$

and these are what 4 becomes if we substitute for ϕ the derivatives

$$\phi', \phi'' \ldots \phi^{(n-2)}.$$

Equation 4 therefore has all of its integrals derivatives of the same function, viz. the integrals are

$$\phi, \frac{d\varphi}{dx}, \frac{d^2\varphi}{dx^2} \cdots \frac{d^{n-2}\varphi}{dx^{n-2}}.$$

Dividing out 4 by p'_{11} and denoting the new coefficients by p'_{22}, p'_{23}, etc., gives the new differential equation

5. $$\frac{d^{n-1}y}{dx^{n-1}} + p'_{22}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p'_{2n}y = 0.$$

The linearly independent integrals of this being

$$\phi, \phi' \ldots \phi^{(n-2)},$$

we can, as before, form a differential equation of the order $n-2$, viz.

6. $$\frac{d^{n-2}y}{dx^{n-2}} + p'_{33}\frac{d^{n-3}y}{dx^{n-3}} + \ldots + p'_{3n}y = 0,$$

where $p'_{33} = p'_{23} \div p'_{22}$, etc., the integrals of which are found just as before to be

$$\phi, \frac{d\varphi}{dx}, \frac{d^2\varphi}{dx^2} \cdots \frac{d^{n-3}\varphi}{dx^{n-3}}.$$

Continuing this process we arrive at the equation of the second degree,

$$\frac{d^2y}{dx^2} + p'_{n-1,\,n-1}\frac{dy}{dx} + p'_{n-1,\,n}y = 0,$$

the integrals of which are ϕ and $\dfrac{d\varphi}{dx}$. Reducing this equation to one of the first order we have

$$\frac{dy}{dx} + \frac{p''_{n-1,\,n}}{p''_{n-1,\,n-1}}y = 0,$$

or for brevity,

$$\frac{dy}{dx} + qy = 0,$$

giving

$$y, = \phi, = e^{-\int qdx}.$$

We have thus determined the complete set of integrals of 2 and consequently also of the derived equations 5, 6, etc.

Writing the equation of the second order in the form

7.
$$\frac{d^2y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = 0,$$

we have for its integrals

$$y_1 = e^{-\int q\, dx}, \quad y_2 = -q e^{-\int q\, dx},$$

where, as before, $q = \dfrac{p'_2}{p'_1}$. Substituting these integrals in the equation we obtain the identities

$$q^2 - q' + p_1 q + p_2 = 0,$$
$$- q^3 - q'' + 3qq' + p_1 q^2 - p_1 q' - p_2 q = 0.$$

Multiplying the first of these by $-q$ and subtracting from the second we have

8.
$$q'' + (p_1 - 2q)\, q' = 0,$$

as the condition to be satisfied by the coefficients of 7, in order that it may have the above integrals. The equation is of course only, theoretically, integrable when we know p_1: and in fact it is obvious that p_1 must be known, for even if we had a differential equation in q independent of p_1 it would still be necessary to know p_1 in order to find p_2 from $q, = \dfrac{p'_2}{p'_1}$.

The solution of 8 can be obtained in an infinite number of cases. First make $p_1 - 2q = F(x)$ where F is any functional symbol. We have then

$$\frac{d^2q}{dx^2} + F(x)\, \frac{dq}{dx} = 0,$$

giving

$$q = C \int e^{-\int F(x)\, dx} dx + C'$$

and

$$p_1 = 2q + F(x)$$

$$p_2 = \int p'_1 q\, dx.$$

Similarly we may write $p_1 - 2q = F(q)$ and obtain values of p_1 and p_2. The only practical difficulty in this second assumption is that x will be determined directly as a function of q instead of the converse.

<div style="text-align:center">EXAMPLES:</div>

a.
$$q'' + (p_1 - 2q)\, q' = 0.$$

Make $p_1 = q$ and we have $\quad q'' - qq' = 0;$

assuming $q' = Q$ this becomes $\quad \dfrac{dQ}{dq} - q = 0,$

giving, if we add no constant of integration,

$$Q = \frac{q^2}{2},$$

and finally

$$q = -\frac{2}{x}.$$

We have therefore

$$p_1 = -\frac{2}{x},$$

and

$$p_2 = \frac{2}{x^2}.$$

The differential equation is then

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2}{x^2}y = 0,$$

of which the integrals are x and x^2.

　　b.　Making again $p_1 = q$ we have

$$q'' - qq' = 0$$

or, if q' be replaced by Q, 　　$$\frac{dQ}{dq} - q = 0.$$

　　Integrating and denoting by $\dfrac{C^2}{2}$ the constant of integration, we have

$$\frac{dy}{dx} + \frac{q^2 + C^2}{2},$$

from which follows

$$q = C \tan \frac{Cx}{2},$$

and the equation is readily found to be

$$\frac{d^2y}{dx^2} + C \tan \frac{Cx}{2}\frac{dy}{dx} + \frac{C^2}{2} \sec^2 \frac{Cx}{2} y = 0,$$

the integrals of which are

$$y_1 = \left(\cos \frac{Cx}{2} \right)^{2C}$$

$$y_2 = \frac{dy_1}{dx} = - C \sin \frac{Cx}{2} \left(\cos \frac{Cx}{2} \right)^{2C-1}$$

　　c.　Making $p_1 = 2q$ the equation is

$$\frac{d^2y}{dx^2} + 2C(1 - Cx)\frac{dy}{dx} + C^3 x (Cx - 2) y = 0,$$

of which the integrals are　　$$y_1 = e^{Cx\left(\frac{Cx}{2} - 1\right)},$$

$$y_2 = \frac{dy_1}{dx} = C(Cx - 1) e^{Cx\left(\frac{Cx}{2} - 1\right)}.$$

.*d.* Making $p_1 - 2q = F(x)$, assume $F(x) = \dfrac{1}{x}$: then $q = C \log x + C'$, and the differential equation is

$$\frac{d^2y}{dx^2} + \left(2C \log x + \frac{1}{x}\right)\frac{dy}{dx} + \left((C \log x)^2 + \frac{C \log x}{x} + \frac{C}{x}\right)y = 0,$$

of which the integrals are

$$y_1 = e^{-(Cx \log x - Cx)}$$

$$y_2 = \frac{dy_1}{dx} = -C \log x \, e^{-(Cx \log x - Cx)}.$$

e. If we write $F(x) = -\dfrac{1}{x}$, we find

$$\frac{d^2y}{dx^2} + \left(Cx^2 - \frac{1}{x}\right)\frac{dy}{dx} + \left(\frac{C^2x^4}{4} + \frac{Cx}{2}\right)y = 0$$

$$y_1 = e^{-\frac{Cx^3}{6}}$$

$$y_2 = \frac{dy_1}{dx} = -\frac{Cx^2}{2}e^{-\frac{Cx^3}{6}}.$$

f. For $F(x) = \alpha$ (α a constant), we have

$$\frac{d^2y}{dx^2} + \left(\alpha - \frac{2C}{\alpha}e^{-\alpha x}\right)\frac{dy}{dx} + \frac{C^2}{\alpha^2}e^{-2\alpha x}y = 0$$

$$y_1 = e^{-\frac{C}{\alpha^2}e^{-\alpha x}}$$

$$y_2 = \frac{dy_1}{dx} = \frac{C}{\alpha}e^{-\alpha x}e^{-\frac{C}{\alpha^2}e^{-\alpha x}}.$$

Writing now as above $\dfrac{p''_{n-n, n}}{p''_{n-1, n-1}} = q$ we have $q'' + (p'_{n-1, n-1} - 2q)\, q' = 0$, which determines in the manner indicated q and $p'_{n-1, n-1}$. Substitute in the equations

$$\frac{d^2y}{dx^2} + p'_{n-1, n-1}\frac{dy}{dx} + p'_{n-1, n}\quad y = 0$$

$$\frac{d^3y}{dx^3} + p'_{n-1, n-1}\frac{d^2y}{dx^2} + p'_{n-1, n}\frac{dy}{dx} = 0,$$

the value $y = e^{-\int q\, dx}$ and we have

$$q^2 - q' - p_{n-1, n-1}\, q + p_{n-1, n} = 0,$$

$$-q^3 + 3qq' - q'' + p'_{n-1, n-1}(q^2 - q') - p'_{n-1, n}\, q = 0.$$

In these $p'_{n-1, n-1}$ is already known, so the first one suffices to determine $p'_{n-1, n}$. Going one step further back to the equations

$$\frac{d^3y}{dx^3} + p'_{n-2, n-2}\frac{d^2y}{dx^2} + p'_{n-2, n-1}\frac{dy}{dx} + p'_{n-2, n}\quad y = 0,$$

$$\frac{d^4y}{dx^4} + p'_{n-2, n-2}\frac{d^3y}{dx^3} + p'_{n-2, n-1}\frac{d^2y}{dx^2} + p'_{n-2, n}\frac{dy}{dx} = 0,$$

$$\frac{d^5y}{dx^5} + p'_{n-2, n-2}\frac{d^4y}{dx^4} + p'_{n-2, n-1}\frac{d^3y}{dx^3} + p'_{n-2, n}\frac{d^2y}{dx^2} = 0,$$

we have for y the same value, viz., $y = e^{-\int q\,dx}, = \phi$, where q is known, therefore for $p'_{n-2,\,n-2}$, $p'_{n-2,\,n-1}$, $p'_{n-2,\,n}$ we have

$$p'_{n-2,\,n-2} = - \begin{vmatrix} \phi''' & \phi' & \phi \\ \phi^{iv} & \phi'' & \phi' \\ \phi^{v} & \phi''' & \phi'' \end{vmatrix} \div$$

$$p'_{n-2,\,n-1} = - \begin{vmatrix} \phi'' & \phi''' & \phi \\ \phi''' & \phi^{iv} & \phi' \\ \phi^{iv} & \phi^{v} & \phi'' \end{vmatrix} \div \quad \text{Denom.} = \begin{vmatrix} \phi'' & \phi' & \phi \\ \phi''' & \phi'' & \phi' \\ \phi^{iv} & \phi''' & \phi'' \end{vmatrix}.$$

$$p'_{n-2,\,n} = - \begin{vmatrix} \phi'' & \phi' & \phi''' \\ \phi''' & \phi'' & \phi^{iv} \\ \phi^{iv} & \phi''' & \phi^{v} \end{vmatrix} \div$$

And so the process is to be continued back to the equation of the nth order, of which the general coefficient will be

$$p_{1i} = - \begin{vmatrix} \overset{1}{\phi^{(n-1)}} \ldots & \overset{2}{\phi^{(n-2)}} \ldots & \overset{i}{\phi^{(n)}} & \ldots \phi \\ \phi^{(n)} \ldots & \phi^{(n-1)} \ldots & \phi^{(n+1)} & \ldots \phi' \\ \vdots & \vdots & \vdots & \vdots \\ \phi^{(2n-2)} \ldots & \phi^{(2n-3)} \ldots & \phi^{(2n-1)} & \ldots \phi^{(n-1)} \end{vmatrix} \div$$

$$\text{Denom.} = \begin{vmatrix} \phi^{(n-1)} \ldots & \phi^{(n-2)} & \ldots \phi \\ \phi^{(n)} \ldots & \phi^{(n-1)} & \ldots \phi' \\ \vdots & \vdots & \vdots \\ \phi^{(2n-2)} \ldots & \phi^{(2n-3)} & \ldots \phi^{(n-1)} \end{vmatrix}.$$

The only thing at all arbitrary consists in the choice of a value for $p'_{n-1,\,n-1} - 2q$ in order that the equation

$$q'' + (p'_{n-1,\,n-1} - 2q)\,q'$$

may be integrated. Parting from the equation of the second order

$$\frac{d^2 y}{dx^2} + p'_{n-1,\,n-1}\,\frac{dy}{dx} + p'_{n-1,\,n}\,y = 0,$$

all the coefficients p'_{ij} up to p_{1k} are determined by the solution of sets of linear equations, the coefficients of which are known functions of the known quantity q.

Assume now that the coefficients of the equation of the nth order, viz.,

$$p_{11},\ p_{12},\ p_{13},\ \cdots p_{1n}$$

are simply periodic functions of the first kind, having ω for their common period.

It is known[*] that such an equation possesses at least one integral which is a periodic function of the second kind, and that the multiplier of this function is a root of the fundamental equation. If we choose, as we may,[†] y_1 for this integral, say $y_1 = \phi$, where $\phi(x + \omega) = \varepsilon \phi(x)$, then

$$\frac{d\varphi}{dx}, \frac{d^2\varphi}{dx^2} \cdots \frac{d^{n-1}\varphi}{dx^{n-1}}$$

will all have the same multiplier as ϕ, so that the roots of the fundamental equation will all be equal and be equal to ε.

It is easy to see that from the form of the integrals of our equation that $y_1, = \phi$, actually is the integral which is periodic of the second kind. If it were not such an integral it would have to be, as is well known, of the form

$$y_1 = \psi_{11} + x\psi_{12} + x^2\psi_{13} + \ldots + x^{n-1}\psi_{1n},$$

where ψ_{ij} is a periodic function of the second kind having ω for a period and ε for a multiplier. All the derivatives of y_1 would now be of the same form as y_1, and consequently no integral of the equation could be a periodic function of the second kind having ω for period and ε for multiplier. But there must be *one* such integral: therefore it follows obviously that y_1 is this integral.

We have now

$$q'' + (p'_{n-1,\,n-1} - 2q)\, q' = 0$$

and

$$y_1 = \phi, = e^{-\int q\, dx},$$

therefore

$$\frac{1}{\varphi} \frac{d\varphi}{dx} = -q\,;$$

changing x into $x + \omega$, the left-hand side of this equation remains unaltered, and therefore q is a periodic function of the first kind having ω for period. Writing then

$$p'_{n-1,\,n-1} - 2q = F(x)$$

it follows that $F(x)$ must also be a periodic function of the first kind, having ω for period, since, from the manner of formation of $p'_{n-1,\,n-1}$ this quantity is such a function. Assuming then that

$$p_{11}, \; p_{12} \cdots p_{1n}$$

are periodic functions of the first kind having ω for period, and forming by the method above indicated the equation

α.

$$q'' + (p'_{n-1,\,n-1} - 2q)\, q' = 0$$

it is only necessary to assume for

$$p'_{n-1,\,n-1} - 2q$$

[*] G. Floquet: Théorie des équations différentielles linéaires à coefficients périodiques. Annales de l'École Normale Supérieure, 1883.

[†] *Vide* Floquet's memoir cited above, §8.

a periodic function of the first kind having the same period, and then determine q, and consequently $p'_{n-1,\,n-1}$ from α, and $p'_{n-1,\,n}$ from

$$q^2 - q' - p'_{n-1,\,n-1}\ q + p'_{n-1,\,n} = 0.$$

The remaining coefficients p'_{ij} and finally p_{1k} will then be determined as above. It is easy to see that the multiplier ε is equal to unity. We have in fact

$$\phi(x) = e^{-\int q(x)\,dx}$$

$$\phi(x + \omega) = \varepsilon\phi(x) = e^{-\int q(x)\,dx}$$

since $$q(x + \omega) = q(x);$$

the right-hand members of these two equations being equal, the left-hand members are also equal, and consequently $\varepsilon = 1$.

It is easy to see that a similar result holds in the case where the coefficients p_{11}, $p_{12} \ldots p_{1n}$ are doubly-periodic coefficients of the first kind having ω and ω' for periods.

If $\phi(x)$ is an integral, doubly-periodic of the second kind, and having ε and ε' for multipliers, *i. e.* if

$$\phi(x + \omega) = \varepsilon\phi(x),$$

$$\phi(x + \omega') = \varepsilon'\phi(x);$$

we have $$\phi(x) = e^{-\int q\,dx},$$

where $q(x + \omega) = q(x + \omega') = q(x)$. And so just as before it is seen that $\varepsilon = \varepsilon' = 1$.

Differentiating twice the first of equations 3, and subtracting the result from the third, we have, since

$$p'_{11} y^{(n)} \quad + p'_{12} y^{(n-1)} + \ldots + p'_{1n} y' = 0,$$

9.
$$p''_{11} y^{(n-1)} + p''_{12} y^{(n-2)} + \ldots + p''_{1n} y = 0.$$

This has for integrals $$\phi,\ \frac{d\varphi}{dx},\ \frac{d^2\varphi}{dx^2} \cdots \frac{d^{n-3}\varphi}{dx^{n-3}},\ z$$

where z has to be determined; this of course is easily done, since, knowing $n - 2$ of the integrals of an equation of the $(n - 1)^{\text{st}}$ order we can reduce the equation to one of the first order, and so determine the remaining integral.

Again, differentiating the first of equations 3 three times, and subtracting from the fourth, we have, after some simple reductions,

10.
$$p'''_{11} y^{(n-1)} + p'''_{12} y^{(n-2)} + \ldots + p'''_{1n} y = 0,$$

of which we know $n - 3$ integrals, viz.

$$\phi,\ \frac{d\varphi}{dx},\ \frac{d^2\varphi}{dx^2},\ \ldots\ \frac{d^{n-4}\varphi}{dx^{n-4}},$$

and which can therefore be reduced to an equation of the second order, which

on being integrated will enable us to find the remaining two integrals of the equation in question.

Similarly we find the equation

$$p_{11}^{iv} y^{(n-1)} + p_{12}^{iv} y^{(n-2)} \ldots + p_{1n}^{iv} y = 0$$

of which

$$\varphi, \frac{d\varphi}{dx}, \frac{d^2\varphi}{dx^2} \cdots \frac{d^{n-5}\varphi}{dx^{n-5}}$$

are integrals, and which can consequently be reduced to an equation of the third order; and

$$p_{11}^{v} y^{(n-1)} + p_{12}^{v} y^{(n-2)} + \ldots + p_{1n}^{v} y = 0$$

of which

$$\varphi, \frac{d\varphi}{dx}, \frac{d^2\varphi}{dx^2} \cdots \frac{d^{n-6}\varphi}{dx^{n-6}},$$

and which can consequently be reduced to an equation of the fourth order.

Finally we come to the equation

$$p_{11}^{(n-1)} y^{(n-1)} + p_{12}^{(n-1)} y^{(n-2)} + \ldots + p_{1n}^{(n-1)} y = 0,$$

of which φ is the only known integral, and which can therefore be only reduced to an equation of the order $n - 2$. The order of these equations may in turn be reduced by unity, and so new sets of equations will arise. The consideration of these equations will, however, be deferred until later.

II.

Equations whose integrals are

$$
\begin{array}{cccc}
y_1 & y_2 & \cdots & y_a \\[6pt]
\dfrac{dy_1}{dx} & \dfrac{dy_2}{dx} & \cdots & \dfrac{dy_a}{dx} \\[6pt]
\dfrac{d^2y_1}{dx^2} & \dfrac{d^2y_2}{dx^2} & \cdots & \dfrac{d^2y_a}{dx^2} \\[4pt]
\vdots & \vdots & & \vdots \\[4pt]
\dfrac{d^{\lambda_1-1}y_1}{dx^{\lambda_1}} & \dfrac{d^{\lambda_2-1}y_2}{dx^{\lambda_2}} & \cdots & \dfrac{d^{\lambda_a-1}y_a}{dx^{\lambda_a}}
\end{array}
$$

where $\lambda_1 + \lambda_2 + \ldots \lambda_a = n$.

We will suppose the quantities $\lambda_1, \lambda_2 \ldots \lambda_a$ arranged in descending order of magnitude, or, at least, in case any of them are equal, in such an order that no one shall be greater than any preceding one. The differential equation is

1. $$\frac{d^n y}{dx^n} + p_{11} \frac{dy^{n-1}}{dx^{n-1}} + p_{12} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{1n} y = 0.$$

Taking the integral y_1 we have the system

$$\Lambda_1 \quad \begin{aligned} \frac{d^n y_1}{dx^n} &+ p_{11}\frac{d^{n-1}y_1}{dx^{n-1}} + p_{12}\frac{d^{n-2}y_1}{dx^{n-1}} + \ldots + p_{1n}y_1 = 0, \\ \frac{d^{n+1}y_1}{dx^{n+1}} &+ p_{11}\frac{d^n y_1}{dx^n} + p_{12}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + p_{1n}\frac{dy_1}{dx} = 0, \\ &\cdots \cdots \cdots \cdots \cdots \\ \frac{d^{n+\lambda_1-1}y_1}{dx^{n+\lambda_1-1}} &+ p_{11}\frac{d^{n+\lambda_1-2}y_1}{dx^{n+\lambda_1-2}} + p_{12}\frac{d^{n+\lambda_1-3}y_1}{dx^{n+\lambda_1-3}} + \ldots + p_{1n}\frac{d^{\lambda_1-1}y_1}{dx^{\lambda_1-1}} = 0. \end{aligned}$$

From the first and second of these we obtain as before

$$(1_1) \qquad p'_{11}\frac{d^{n-1}y_1}{dx^{n-1}} + p'_{12}\frac{d^{n-2}y_1}{dx^{n-2}} + \ldots + p'_{1n}y = 0,$$

from the second and third we have

$$(2_1) \qquad p'_{11}\frac{d^n y_1}{dx^n} + p'_{12}\frac{d^{n-1}y_1}{dx^{n-1}} + \ldots + p'_{1n}\frac{dy}{dx} = 0,$$

from the third and fourth

$$(3_1) \qquad p'_{11}\frac{d^{n+1}y_1}{dx^{n+1}} + p'_{12}\frac{d^n y}{dx^n} + \ldots + p'_{1n}\frac{d^2 y}{dx^2} = 0,$$

and finally

$$p'_{11}\frac{d^{n+\lambda_1-2}y_1}{dx^{n+\lambda_1-2}} + p'_{12}\frac{d^{n+\lambda_1-3}y_1}{dx^{n+\lambda_1-3}} + \ldots + p'_{1n}\frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}} = 0.$$

These last equations are merely what the first becomes on replacing y_1 by $\frac{dy_1}{dx}, \frac{d^2 y_1}{dx^2} \ldots \frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}}$. These quantities therefore are all integrals of the equation

$$(1'_1) \qquad p'_{11}y^{(n-1)} + p'_{12}y^{(n-2)} + \ldots + p'_{1n}y = 0.$$

A set of equations similar to Λ_1 are to be formed for each of the integrals $y_2, y_3 \ldots y_a$; call the general set Λ_i. this is

$$\Lambda_i \quad \begin{aligned} \frac{d^n y_i}{dx^n} &+ p_{11}\frac{d^{n-1}y_i}{dx^{n-1}} + p_{12}\frac{d^{n-2}y_i}{dx^{n-2}} + \ldots + p_{1n}y_i = 0. \\ \frac{d^{n+1}y_i}{dx^{n+1}} &+ p_{11}\frac{d^n y_i}{dx^n} + p_{12}\frac{d^{n-1}y_i}{dx^{n-1}} + \ldots + p_{1n}\frac{dy_i}{dx} = 0. \\ &\cdots \cdots \cdots \cdots \cdots \\ \frac{d^{n+\lambda_i-1}y_i}{dx^{n+\lambda_i-1}} &+ p_{11}\frac{d^{n+\lambda_i-2}y_i}{dx^{n+\lambda_i-2}} + p_{12}\frac{d^{n+\lambda_i-3}y_i}{dx^{n+\lambda_i-3}} + \ldots + p_{1n}\frac{d^{\lambda_i-1}y_i}{dx^{\lambda_i-1}} = 0. \end{aligned}$$

From these, as before, we form the system

$$(1_i) \qquad p'_{11}y_i^{n-1} + p'_{12}y_i^{n-2} + \ldots + p'_{1n}y_i = 0,$$
$$(2_i) \qquad p'_{11}y_i^{n} + p'_{12}y_i^{n-1} + \ldots + p'_{1n}y'_i = 0,$$
$$\cdots \cdots \cdots \cdots \cdots$$
$$p'_{11}y_i^{n+\lambda_i-2} + p'_{12}y_i^{n+\lambda_i-3} + \ldots + p'_{1n}y_i^{\lambda_i-2} = 0.$$

Referring now to equation (1_1) it is clear from equation (1_i) that $y_2, y_3 \ldots y_a$ are all integrals, and, since $\lambda_2, \lambda_3 \ldots \lambda_a$ are none of them greater than λ_1 and no element in this series is greater than any preceding one, it also follows that the derivatives of $y_2, y_3 \ldots y_a$ up to the orders $\lambda_2 - 2, \lambda_3 - 2 \ldots \lambda_a - 2$ are the remaining integrals. That is, the complete set of integrals of $(1_1')$ is

$$y_1, \; \frac{dy_1}{dx}, \; \frac{d^2y_1}{dx^2}, \; \ldots \; \frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}},$$

$$y_2, \; \frac{dy_2}{dx}, \; \frac{d^2y_2}{dx^2}, \; \ldots \; \frac{d^{\lambda_2-2}y_2}{dx^{\lambda_1-1}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$y_a, \; \frac{dy_a}{dx}, \; \frac{d^2y_a}{dx^2}, \; \ldots \; \frac{d^{\lambda_a-2}y_a}{dx^{\lambda_a-1}},$$

or taking the general set of integrals,

$$(P_i) \quad \begin{matrix} (1_i) \\ (2_i) \end{matrix} \quad \begin{matrix} p_{11}'y_i^{n-1} & + p_{12}'y_i^{n-2} & + \ldots + p_{1n}'y_i & = 0, \\ p_{11}'y_i^{n} & + p_{12}'y_i^{n-1} & + \ldots + p_{1n}'y_i' & = 0, \end{matrix}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$p_{11}'y_i^{n+\lambda_i-2} + p_{12}'y_i^{n+\lambda_i-3} + \ldots + p_{1n}'y_i^{\lambda_i-2} = 0.$$

The first of these is also satisfied by

$$y_1, \; y_2 \ldots y_{i-1}, \; y_{i+1} \ldots y_a.$$

It is clear from what precedes that the following also satisfy (1_i),

$$\frac{dy_1}{dx}, \quad \frac{d^2y_1}{dx^2} \; \cdots \; \frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}},$$

$$\frac{dy_2}{dx}, \quad \frac{d^2y_2}{dx^2} \; \cdots \; \frac{d^{\lambda_2-2}y_2}{dx^{\lambda_2-2}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{dy_{i-1}}{dx}, \quad \frac{d^2y_{i-1}}{dx^2} \cdots \; \frac{d^{\lambda_{i-1}-2}y}{dx^{\lambda_{i-1}-2}}.$$

The quantities

$$\frac{dy_{i+1}}{dx}, \quad \frac{d^2y_{i+1}}{dx^2} \cdots \; \frac{d^{\lambda_{i+1}-2}y_{i+1}}{dx^{\lambda_{i+1}-1}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{dy_a}{dx}, \quad \frac{d^2y_a}{dx^2} \; \cdots \; \frac{d^{\lambda_a-2}y_a}{dx^{\lambda_a-1}},$$

will obviously also satisfy (1_i). We have thus the same complete set of integrals as before obtained, viz. the known integrals of the derived equation

$$\frac{d^{n-1}y}{dx^{n-1}} + p_{22}^{(1)}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + p_{2n}^{(1)}y = 0,$$

where $p_{2i}^{(1)} = p_{1i}' \div p_{11}'$, are

$$y_1, \frac{dy_1}{dx}, \frac{d^2y_1}{dx^2}, \ldots \frac{d^{\lambda_1-2}y_1}{dx^{\lambda_1-2}},$$

$$y_2, \frac{dy_2}{dx}, \frac{d^2y_2}{dx^2}, \ldots \frac{d^{\lambda_2-2}y_2}{dx^{\lambda_2-2}},$$

$$y_a, \frac{dy_a}{dx}, \frac{d^2y_a}{dx}, \ldots \frac{d^{\lambda_a-2}y_a}{dx^{\lambda_a-2}}.$$

We have in all these $\lambda_1 + \lambda_2 + \ldots + \lambda_a - a, = n - a$, known integrals, *i. e.* there remain $a - 1$ integrals to be determined. It will of course be noticed that if for any value of i $\lambda_i = 1$ the corresponding function y_i is not an integral of the derived equation. Referring now to the system (P_i): writing as above

$$\frac{p_{12}'}{p_{11}'} = p_{22}^{(1)}, \frac{p_{13}'}{p_{11}'} = p_{23}^{(1)}, \text{ etc.}$$

this is

$$(P_i)_1 \qquad \begin{aligned} y_i^{(n-1)} &+ p_{22}^{(1)}y_i^{(n-2)} &+ \ldots + p_{2n}^{(1)}y_i &= 0, \\ y_i^{(n)} &+ p_{22}^{(1)}y_i^{(n-1)} &+ \ldots + p_{2n}^{(1)}y_i' &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_i^{(n+\lambda_i-2)} &+ p_{22}^{(1)}y_i^{(n+\lambda_i-3)} &+ \ldots + p_{2n}^{(1)}y_i^{(\lambda_i-2)} &= 0. \end{aligned}$$

We have now as is easily seen, for the integrals of the equation

$$P^{(1)} \qquad p_{22}^{(1)\prime}y^{(n-2)} + p_{23}^{(1)\prime}y^{(n-3)} + \ldots + p_{2n}^{(1)\prime}y = 0,$$

the following system,

$$y_1 \quad \frac{dy_1}{dx} \quad \frac{d^2y_1}{dx^2} \cdots \frac{d^{\lambda_1-3}y_1}{dx^{\lambda_1-3}},$$

$$y_2 \quad \frac{dy_2}{dx} \quad \frac{d^2y_2}{dx^2} \cdots \frac{d^{\lambda_2-3}y_2}{dx^{\lambda_2-3}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$y_i \quad \frac{dy_i}{dx} \quad \frac{d^2y_i}{dx^2} \cdots \frac{d^{\lambda_i-3}y_i}{dx^{\lambda_i-3}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$y_a \quad \frac{dy_a}{dx} \quad \frac{d^2y_a}{dx^2} \cdots \frac{d^{\lambda_a-3}y_a}{dx^{\lambda_a-3}}.$$

We have thus $n - 2a$ integrals of the equation of order $n - 2$, leaving $2(a-1)$ integrals to be determined. Again we can form a new equation of the degree $n - 3$ of which we know $n - 3a$ integrals and $3(a-1)$ to be determined. This process will be continued until $n - ka$ becomes either equal to or less than a. If $\lambda_1 = \lambda_2 \ldots \lambda_a$ then we shall have $n - ka = a$ as the number of integrals of the equation which cannot again be reduced. These will be $y_1, y_2 \ldots y_a$. If